Skin friction on a strip of finite width moving parallel to its length

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(*Received* 24 *June* 1957)

Summary

A flat strip, of infinitesimal thickness and infinite length, is located in a viscous incompressible fluid, and both the strip and fluid are at rest initially. The strip is abruptly set into steady motion parallel to its length. An unsteady uni-directional flow of the fluid results, and there is a variable skin friction on the strip which must be overcome to maintain its velocity. In the early stages of motion, the skin friction is large, with a local behaviour which resembles that for a flat strip of infinite width. The skin friction near each edge of the strip can be more accurately represented by referring to the semi-infinite configuration that is realized on displacement of the other edge to infinity. At this stage, the results depend on separate consideration of the two limiting configurations, and the way to further improvements is not clearly delineated. The object of this paper is to provide a formulation which contains the preceding information and allows a systematic evaluation of all additional refinements. Thus, it is shown that the total skin friction on a strip of width $2a$, moving with velocity V in a fluid whose coefficients of viscosity and kinematic viscosity are μ, ν, takes the form

$$D = 2\mu V\left[\frac{2a}{\sqrt{(\pi\nu t)}}+1-\frac{2}{\pi}\int_1^\infty \frac{1}{u^2}\sqrt{\left(\frac{u-1}{u+1}\right)}\operatorname{erfc}\left(\frac{au}{\sqrt{(\nu t)}}\right)du\right].$$

Here the first two terms stem from the infinite and semi-infinite strip distributions, and the integral, containing a complementary error function, furnishes all corrections of the lowest exponential order, in the initial stages of motion, when $a/\sqrt{(\nu t)} \gg 1$. The calculation is based on a new integral equation which makes explicit the effect of an isolated edge, and by iteration provides the interaction effects between edges at a finite separation.

1. Introduction

A simple class of viscous incompressible fluid motions is that produced by the forced motion of a solid parallel to its infinite length. If the solid is instantaneously set in motion with constant velocity, and has a uniform aspect along its length, the resulting fluid motion is uni-directional, although unsteady in time. The fluid velocity is parallel to and uniform in the direction of motion of the solid, and hence described by a linear partial differential equation with the time and transverse coordinates as independent variables*. Rayleigh (1911) analysed the simplest configuration, a flat strip of infinite width, for which the skin friction is uniform and has the magnitude $\mu V/\sqrt{(\pi\nu t)}$ per unit area (of each face). The strip of semi-infinite width and the more general wedge shape have been studied by Howarth (1950), Sowerby (1951), and Hasimoto (1951 a, b), and from these calculations the edge effect can be isolated. A characteristic dimension is introduced by the strip of finite width, say $2a$, and the state of affairs depends on whether $a/\sqrt{(\nu t)}$ is large or small compared with unity. In the former case, the fluid motion is confined to a boundary layer with small thickness relative to the strip width, and the total skin friction is given approximately by supplementing the Rayleigh contribution with those of the isolated edges, whence

$$D = \mu V\left[\frac{4a}{\sqrt{(\pi\nu t)}}+2\right] \quad (a/\sqrt{(\nu t)} \gg 1). \tag{1}$$

This result, obtained by Howarth (1950) and Hasimoto (1951 a, b) (Davies (1950) also considers the flat strip, presenting a formal series solution which does not lend itself to analytic approximation for small t), omits any interaction between the strip edges, and it is such effects that we propose to calculate. It turns out that their magnitude is rather small, in the domain where the skin friction may be analysed by the preceding scheme, for the corrections to (1) are of exponential order; thus

$$D = \mu V\left[\frac{4a}{\sqrt{(\pi\nu t)}}+2-\frac{1}{2\pi}\left(\frac{\sqrt{(\nu t)}}{a}\right)^4 e^{-a^2/\nu t}+O\left\{\left(\frac{\sqrt{(\nu t)}}{a}\right)^6 e^{-a^2/\nu t}\right\}\right] \quad (a/\sqrt{(\nu t)} \gg 1). \tag{2}$$

The developments of the skin friction for the strip and the (smooth) circular cylinder show a quite different form after the first two terms, since the latter is, according to Batchelor (1952),

$$D = \mu V\left[\frac{2\pi a}{\sqrt{(\pi\nu t)}}+\pi-\frac{\sqrt{(\pi\nu t)}}{2a}+\frac{\pi\nu t}{4a^2}+\ldots\right] \quad (a/\sqrt{(\nu t)} \gg 1), \tag{3}$$

where a denotes the cylinder radius. For the later times, $\sqrt{(\nu t)}/a \gg 1$, Hasimoto (1954) and Batchelor have obtained other developments of the skin friction in the two cases; there is a smooth transition to the approximate expression (1) at $a/\sqrt{(\nu t)} = 1$. Hasimoto (1954, 1955) has recently generalized (3) so as to apply to a smooth general cylinder, but the corresponding accuracy is not yet attained for a polygonal cross-section.

The calculation to be described makes use of a double transform operation, one being a Laplace transform with respect to the time and the other a complex Fourier transform with respect to the (transverse) coordinate in the plane of the strip. After applying the Laplace transform to the differential equation for the fluid velocity, the resulting boundary

* An equivalent heat conduction problem relates to the variable temperature distribution in a uniform medium which is initially at zero temperature, and in which an internal boundary is maintained at a constant temperature. Here the heat flux at the boundary is analogous to the skin friction in the fluid motion problem.

value problem is formulated (§ 2) in terms of Green's functions and integral equations. The subsequent analysis is effectively carried out with complex Fourier transform and elementary function theoretic arguments, yielding finally a new integral equation (§ 3) which is well adapted to the determination of the skin friction (§ 4) in the initial phases of the motion. It is noteworthy that the inverse Laplace transform back to the time coordinate can be performed without approximation at any stage of iteration for the integral equation.

Similar Fourier transform techniques can be employed to analyse the scattering of sound or light waves incident on a strip, and the results invite comparison with those in the problem at hand. In the time-harmonic two-dimensional situation, with primary plane waves, the scattering cross-section at normal incidence relates to an integrated strip distribution and is thus analogous to the total skin friction. For comparison purposes, consider the skin friction (3) and the leading terms in the cross-section σ at wavelengths λ short compared with the strip width $2a$,

$$\sigma = 4a - \frac{4a}{\sqrt{\pi}}\left(\frac{\lambda}{2\pi a}\right)^{3/2}\cos\left(\frac{4\pi a}{\lambda}-\frac{\pi}{4}\right)+\frac{2a}{\pi}\left(\frac{\lambda}{2\pi a}\right)^{2}\cos\frac{8\pi a}{\lambda} \quad (\lambda/a \leqslant 1), \tag{4}$$

when the total wave function vanishes at the strip. The counterpart of the Rayleigh term in the skin friction is the geometrical term of the cross-section, and whereas the next contribution to the skin friction arises from isolated edge effects, these do not contribute to the cross-section. Succeeding terms of the developments, which account for interaction between the strip edges, are of exponential form in the skin friction and of trigonometric form in the cross-section, reflecting the diffusion and propagation character of the problems.

Carrier (1955) has also studied the boundary layer aspect of solutions to certain types of integral equation, where reference is made to a semi-infinite configuration for the purpose of characterizing the edge behaviour. The solution to an inhomogeneous finite-range integral equation (cf. (21)) is given, neglecting interaction between the edges, and utilized in analysis of an oscillating plate in a viscous fluid.

2. Formulation

Let the strip be located in the (x, z)-plane,

$$-a \leqslant x \leqslant +a, \qquad -\infty < z < +\infty,$$

and set into motion at time $t = 0$ with steady velocity V along the z-direction. The viscous fluid begins a parallel motion, with velocity $v_z(x, y, t)$ which satisfies the equation

$$\frac{\partial v_z}{\partial t} = \nu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}\right) \tag{5}$$

(ν is the kinematic viscosity) and automatically complies with the equation of continuity, by virtue of translational invariance. The same differential equation applies to the velocity ratio

$$\psi(x, y, t) = v_z/V, \tag{6}$$

which is further subject to the initial and boundary conditions

$$\psi = 0 \quad \text{at } t = 0, \tag{7}$$

$$\left.\begin{aligned} \psi &= 1 \quad \text{at } y = 0 \quad \text{and } |x| < a, \\ \psi &\to 0 \quad \text{as } x^2+y^2 \to \infty, \end{aligned}\right\} \quad (t > 0) \tag{8}$$

and the symmetry condition

$$\psi(x,y,t) = \psi(x,-y,t). \tag{9}$$

If

$$\bar{\psi}(x,y,p) = p\int_0^\infty e^{-pt}\psi(x,y,t)\,dt \tag{10}$$

denotes a quantity proportional to the Laplace transform of $\psi(x,y,t)$, then, using (5), (6), (7), it follows that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right)\bar{\psi}(x,y,p) = 0 \tag{11}$$

and

$$\left.\begin{aligned} \bar{\psi} &= 1 \quad \text{at } y = 0 \quad \text{and } |x| < a, \\ \bar{\psi} &\to 0 \quad \text{as } x^2+y^2 \to \infty, \end{aligned}\right\} \tag{12}$$

where

$$k^2 = p/\nu. \tag{13}$$

The boundary value problem stated in (11), (12), is advantageously reformulated with the help of solutions to the associated inhomogeneous differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right)G(x,y;\,x',y') = -\delta(x-x')\,\delta(y-y') \quad (\mathscr{R}\{k\} > 0). \tag{14}$$

A fundamental solution of this equation with logarithmic singularity at the 'source' point $x = x'$, $y = y'$, and a null value at infinity is the Green's function

$$G_0(x,y;\,x',y') = (2\pi)^{-1}K_0(k\surd\{(x-x')^2+(y-y')^2\}), \tag{15}$$

where K_0 denotes the zero order Hankel function of imaginary argument, namely

$$K_0(\zeta) = \tfrac{1}{2}\pi i H_0^{(1)}(\zeta e^{i\pi/2}). \tag{16}$$

In terms of the source function G_0, a solution of the differential equation (11) which vanishes at infinity and is symmetric with respect to the plane of the strip can be represented by

$$\bar{\psi}(x,y,p) = \frac{1}{2\pi}\int_{-a}^{a} K_0(k\surd\{(x-x')^2+y^2\})g(x')\,dx'. \tag{17}$$

Placing $y = 0$, and writing

$$\bar{\psi}(x,0,p) = f(x), \tag{18}$$

we have

$$f(x) = \frac{1}{2\pi}\int_{-a}^{a} K_0(k|x-x'|)g(x')\,dx', \tag{19}$$

and from the condition (12), which gives $f(x) = 1$ for $|x| < a$, there follows an integral equation for the (vorticity distribution function)

$$g(x) = -2\left(\frac{\partial\bar{\psi}}{\partial y}\right)_{y=0} \tag{20}$$

on the strip. The boundary value problem (11), (12), is thus linked with the solution of a one-dimensional integral equation,

$$1 = \frac{1}{2\pi}\int_{-a}^{a} K_0(k|x-x'|)g(x')\,dx' \quad (|x| < a), \tag{21}$$

and a knowledge of $g(x)$ determines $\bar{\psi}$ at all points off the strip by (17).

Choosing next the Green's function of the half space $y > 0$,

$$G_1(x,y;\,x',y') = (2\pi)^{-1}[K_0(k\surd\{(x-x')^2+(y-y')^2\}) - \\ - K_0(k\surd\{(x-x')^2+(y+y')^2\})], \tag{22}$$

which vanishes at $y = 0$, a representation for ψ therein is,

$$\bar{\psi}(x,y,p) = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_{|x'|<a} K_0(k\surd\{(x-x')^2+y^2\})\,dx' - \\ -\frac{1}{\pi}\frac{\partial}{\partial y}\int_{|x'|>a} K_0(k\surd\{(x-x')^2+y^2\})f(x')\,dx' \quad (y>0), \tag{23}$$

where the boundary value $\bar{\psi}(x,0,p) = f(x) = 1$ for $|x| < a$ is utilized. A further differentiation in (23) with respect to y, and passage to the limit $y = 0$, yields

$$-\tfrac{1}{2}g(x) = -\frac{1}{\pi}\frac{\partial^2}{\partial y^2}\left(\int_{|x'|<a} K_0(k\surd\{(x-x')^2+y^2\})\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k\surd\{(x-x')^2+y^2\})f(x')\,dx'\right)_{y\to 0+} \\ = \frac{1}{\pi}\left(\frac{d^2}{dx^2}-k^2\right)\left(\int_{|x'|<a} K_0(k|x-x'|)\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k|x-x'|)f(x')\,dx'\right), \tag{24}$$

for the K_0-function is a solution of the homogeneous equation (11) when $y' = 0$ and $y \to 0$. Since $\bar{\psi}(x,y,p) = \bar{\psi}(x,-y,p)$, it follows that $(\partial\bar{\psi}/\partial y)_{y=0} = 0$ for $|x| > a$, and the condition $g(x) = 0$ for $|x| > a$ provides an integro-differential equation,

$$\left(\frac{d^2}{dx^2} - k^2\right)\left(\int_{|x'|<a} K_0(k|x-x'|)\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k|x-x'|)f(x')\,dx'\right) = 0 \quad (|x| > a), \tag{25}$$

for the transform $\bar{\psi}(x,0,p) = f(x)$, in the domain $|x| > a$. The solution to the boundary value problem (11), (12), can be based on this integro-differential equation, and the representation (23) for the even function of y, $\bar{\psi}(x,y,p)$, in the half-plane $y > 0$. It is evident that the two formulations given are of a complementary nature, with the primary role accorded to the distribution $g(x)$ on the strip in the former, and to the coplanar distribution $f(x)$ off the strip in the latter; clearly, a knowledge of either function suffices to determine the other.

The skin friction $D(x)$ at the strip is

$$D(x) = -\mu\left(\frac{\partial v_z}{\partial y}\right)_{y=0} = \tfrac{1}{2}\mu V L^{-1}[g(x,p)], \tag{26}$$

where L^{-1} denotes an inverse Laplace transform, namely (recall the additional factor p introduced in (10)),

$$L^{-1}[g(x,p)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{g(x,p)}{p}e^{pt}\,dp \quad (c > 0). \tag{27}$$

In the total skin friction or drag, D, the integral of $D(x)$ over the strip width is involved, and an additional factor of 2 is necessary to account for both faces, whence

$$D = \mu V L^{-1}\left(\int_{-a}^{a} g(x,p)\,dx\right) = \frac{\mu V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{dp}{p}\,e^{pt}\int_{-a}^{a} g(x,p)\,dx. \tag{28}$$

Although the drag is naturally related to $g(x)$, the latter function need not be the primary quantity in an integral equation formulation of the boundary value problem; to calculate D in the early stages of motion, it will in fact be convenient to work from the integro-differential equation for $f(x)$ and obtain $g(x)$ secondarily.

3. Integral equations

The basic integral equations (21) and (25) cannot be solved explicitly in closed form, and, as presently constituted, do not lend themselves to approximate solution, say, by iteration. Our objective is to obtain a new integral equation from (25) which is appropriate to the conditions prevailing at small values of t, and which also permits ready approximation to the skin friction. The idea underlying this transformation is that, for large values of the parameter $a/\surd(\nu t)$, the distributions $f(x)$ in the ranges $x > a$ and $x < -a$ are weakly coupled and exactly describable in the limit of no coupling, which corresponds to the geometrical configuration of a semi-infinite strip.

A brief exploration of the latter problem is in order, and information obtained relative to the behaviour of f and g near the edge will carry over to the finite strip case. The relation (19) provides a convenient starting point, and, if the origin is shifted to one edge, this relation becomes

$$\left.\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{2a} K_0(k|x-x'|)g(x')\,dx' \\ &= 1 \quad \text{for } 0 < x < 2a. \end{aligned}\right\} \tag{29}$$

Whence, letting $a \to \infty$, we obtain

$$\left.\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{\infty} K_0(k|x-x'|)g(x')\,dx' \\ &= 1 \quad \text{for } x > 0 \end{aligned}\right\} \tag{30}$$

as the form appropriate to the semi-infinite strip $0 < x < \infty$. Employing the integral representation

$$K_0(k\sqrt{\{(x-x')^2+(y-y')^2\}}) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\exp\{i\zeta(x-x')-\sqrt{(k^2+\zeta^2)}|y-y'|\}}{\sqrt{(k^2+\zeta^2)}}\,d\zeta, \tag{31}$$

and introducing

$$\bar{f}(\zeta) = \int_{-\infty}^{0} e^{-i\zeta x} f(x)\,dx, \tag{32}$$

$$\bar{g}(\zeta) = \int_{0}^{\infty} e^{-i\zeta x} g(x)\,dx, \tag{33}$$

we then obtain the transform version of (30):

$$\bar{f}(\zeta)+\frac{1}{\epsilon+i\zeta} = \frac{\bar{g}(\zeta)}{2\sqrt{(k^2+\zeta^2)}}. \tag{34}$$

Here ϵ appears by virtue of an exponential attenuation factor introduced when calculating the transform of $f = 1$ in the range $x > 0$. The relation (34) has meaning if there is a common domain of regularity in the complex ζ-plane for the functions occurring therein. According to (30), $f(x)$ contains the factor e^{kx} as $x \to -\infty$, and hence $\bar{f}(\zeta)$ is regular in the half-plane $\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}$; an exponential free behaviour for the strip distribution $g(x)$, as $x \to \infty$, implies that $\bar{g}(\zeta)$ is regular in the half-plane $\mathscr{I}\{\zeta\} < 0$. Thus, the common domain of regularity is a strip parallel to the real axis of the ζ-plane, defined by $0 > \mathscr{I}\{\zeta\} > -\mathscr{R}\{k\} = \mathscr{I}\{-ik\}$. Following the Wiener–Hopf procedure, (34) is rewritten in the form

$$\sqrt{(\zeta+ik)}\bar{f}(\zeta)+\frac{\sqrt{(\zeta+ik)}-\sqrt{(ik+i\epsilon)}}{\epsilon+i\zeta} = -\frac{\sqrt{(ik+i\epsilon)}}{\epsilon+i\zeta}+\frac{\bar{g}(\zeta)}{2\sqrt{(\zeta-ik)}}, \tag{35}$$

where the left-hand side is regular in the (upper) half-plane $\mathscr{I}\{\zeta\} > \mathscr{I}\{-ik\}$, and the right-hand side is regular in the (lower) half-plane $\mathscr{I}\{\zeta\} < 0$. An integral function is thus defined throughout the ζ-plane, and, since each side vanishes at infinity in the respective half-planes (assuming only that $f(x)$ and $g(x)$ are integrable at the edge $x = 0$), this function is zero. Hence,

$$\bar{f}(\zeta) = \frac{i}{\zeta}\left[1-\sqrt{\left(\frac{ik}{\zeta+ik}\right)}\right], \tag{36}$$

in the limit $\epsilon = 0$, while

$$\bar{g}(\zeta) = 2\sqrt{(ik)}\frac{\sqrt{(\zeta-ik)}}{\epsilon+i\zeta}. \tag{37}$$

Inverting, we find

$$\begin{aligned} f(x) &= \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\zeta x}}{\zeta}\left[1-\sqrt{\left(\frac{ik}{\zeta+ik}\right)}\right]d\zeta \\ &= \frac{i}{2\pi}\int_{-i\infty}^{i\infty}\frac{e^{kxu}}{u}\left[1-\sqrt{\left(\frac{1}{1-u}\right)}\,du\right] = \frac{1}{\pi}\int_{1}^{\infty}\frac{e^{kxu}}{u\sqrt{(u-1)}}\,du \quad (x<0), \end{aligned} \tag{38}$$

and

$$\begin{aligned} g(x) &= \frac{\sqrt{(ik)}}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\zeta x}}{\epsilon+i\zeta}\sqrt{(\zeta-ik)}\,d\zeta = -\frac{ik}{\pi}\int_{-i\infty}^{i\infty}\frac{e^{kxu}}{u+\epsilon/k}\sqrt{(u+1)}\,du \\ &= 2k+\frac{2k}{\pi}\int_{1}^{\infty}\frac{e^{-kxu}}{u}\sqrt{(u-1)}\,du \quad (x>0), \end{aligned} \tag{39}$$

with k real. From (38), it appears that

$$\lim_{x\to 0-} f(x) = \frac{1}{\pi}\int_1^\infty \frac{du}{u\sqrt{(u-1)}} = 1, \tag{40}$$

which is the value of f for $x > 0$, and

$$f(x) \sim e^{kx}/(-kx)^{1/2} \quad \text{as } x \to -\infty. \tag{41}$$

According to (39),

$$g(x) \sim (kx)^{-1/2} \quad \text{as } x \to 0+ \tag{42}$$

and

$$g(x) \sim 2k \quad \text{as } x \to \infty. \tag{43}$$

Since

$$1 = \frac{1}{2\pi}\int_{-\infty}^\infty K_0(k|x-x'|)(2k)\,dx' \quad (-\infty < x < \infty), \tag{44}$$

the uniform distribution $g(x) = 2k$ evidently refers to a strip of infinite width. This synopsis of features of the semi-infinite strip configuration is sufficient for the present, and further references are deferred until later.

Turning attention again to the strip of finite width, and choosing (24) as a starting-point for the analysis, we find that the concomitant transform relation becomes

$$\tfrac{1}{2}\bar{g}(\zeta) = \sqrt{(k^2+\zeta^2)}\left[\bar{f}(\zeta) - \left(\frac{e^{i\zeta a}-e^{-i\zeta a}}{i\zeta}\right)\right], \tag{45}$$

where
$$\bar{g}(\zeta) = \int_{-a}^{a} e^{-i\zeta x} g(x)\,dx = \bar{g}(-\zeta), \tag{46}$$

and
$$\bar{f}(\zeta) = \bar{f}_1(\zeta) + \bar{f}_2(\zeta), \tag{47}$$

with
$$\bar{f}_1(\zeta) = \int_a^\infty e^{-i\zeta x} f(x)\,dx, \tag{48}$$

$$\bar{f}_2(\zeta) = \int_{-\infty}^{-a} e^{-i\zeta x} f(x)\,dx. \tag{49}$$

Evidently $\bar{g}(\zeta)$ is regular in the finite part of the ζ-plane, and with the local behaviour $g(x) \sim (k(a\mp x))^{-1/2}$ as $x \to \pm a$, which is characteristic of an isolated edge (see (42)), it follows that

$$\bar{g}(\zeta) \sim e^{\mp i\zeta a}/(\mp i\zeta)^{1/2} \quad \text{as } |\zeta| \to \infty \quad (\mathscr{I}\{\zeta\} \gtrless 0). \tag{50}$$

The exponential decrease of $f(x)$ as $x \to \infty$ implies that $\bar{f}_1(\zeta)$ is regular in the (lower) half-plane $\mathscr{I}\{\zeta\} < \mathscr{R}\{k\}$, and a finite limit for $f(x)$ as $x \to a+0$ (see (40)) yields the asymptotic behaviour

$$\bar{f}_1(\zeta) \sim e^{-i\zeta a}/\zeta \quad \text{as } |\zeta| \to \infty \qquad (\mathscr{I}\{\zeta\} < \mathscr{R}\{k\}) \tag{51}$$

in the half-plane of regularity. By symmetry, f is an even function of x, so that $\bar{f}_2(\zeta) = \bar{f}_1(-\zeta)$, and thus $\bar{f}_2(\zeta)$ is regular in the (upper) half-plane $\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}$, where

$$\bar{f}_2(\zeta) \sim e^{i\zeta a}/\zeta \quad \text{as } |\zeta| \to \infty \qquad (\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}). \tag{52}$$

Accordingly, the transform relation (45) holds in the strip $|\mathscr{I}\{\zeta\}| < \mathscr{R}\{k\}$ and, after the substitutions

$$\bar{f}_1(\zeta) = e^{-i\zeta a}\bar{F}_1(\zeta), \qquad \bar{f}_2(\zeta) = e^{i\zeta a}\bar{F}_2(\zeta), \tag{53}$$

takes the form

$$\tfrac{1}{2}\bar{g}(\zeta) = \sqrt{(k^2+\zeta^2)}[e^{-i\zeta a}\bar{F}_1(\zeta)+e^{i\zeta a}\bar{F}_2(\zeta)+(e^{i\zeta a}-e^{-i\zeta a})/i\zeta]. \quad (54)$$

Multiplying in turn by $e^{i\zeta a}$, $e^{-i\zeta a}$, we get

$$\tfrac{1}{2}\bar{g}(\zeta)e^{i\zeta a} = \sqrt{(k^2+\zeta^2)}[\bar{F}_1(\zeta)+e^{2i\zeta a}\bar{F}_2(\zeta)+(e^{2i\zeta a}-1)/i\zeta], \quad (55)$$

$$\tfrac{1}{2}\bar{g}(\zeta)e^{-i\zeta a} = \sqrt{(k^2+\zeta^2)}[e^{-2i\zeta a}\bar{F}_1(\zeta)+\bar{F}_2(\zeta)+(1-e^{-2i\zeta a})/i\zeta]. \quad (56)$$

On the factorization of the radical,

$$\sqrt{(k^2+\zeta^2)} = \sqrt{\{(\zeta+ik)(\zeta-ik)\}},$$

the first of these equations can be written

$$\tfrac{1}{2}\bar{g}(\zeta)\frac{e^{i\zeta a}}{\sqrt{(\zeta+ik)}} = \sqrt{(\zeta-ik)}\left[\bar{F}_1(\zeta)+e^{2i\zeta a}\bar{F}_2(\zeta)+\left(\frac{e^{2i\zeta a}-1}{i\zeta}\right)\right].$$

The left-hand member is regular in the upper half-plane, and the first term of the right-hand member is regular in the lower half-plane. For each remaining term of the right-hand member, a decomposition may be effected, with the help of Cauchy's integral theorem,

$$w(z) = \frac{1}{2\pi i}\oint\frac{w(\xi)}{\xi-z}\,d\xi,$$

into a sum of functions regular in the upper and lower half-planes, respectively. Collecting all terms regular in the respective overlapping

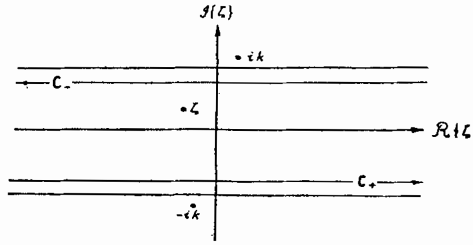

Figure 1.

half-planes, an integral function is defined throughout the ζ-plane, with the following representation in the lower half-plane:

$$\text{I.F.} = \sqrt{(\zeta-ik)}\bar{F}_1(\zeta)+\frac{1}{2\pi i}\int_{C_-}\sqrt{(\xi-ik)}e^{2i\xi a}\bar{F}_2(\xi)\frac{d\xi}{\xi-\zeta}+$$
$$+\frac{1}{2\pi i}\int_{C_-}\sqrt{(\xi-ik)}\left(\frac{e^{2i\xi a}-1}{i\xi}\right)\frac{d\xi}{\xi-\zeta}. \quad (57)$$

Here C_- is a path of integration conducted in the strip $|\mathscr{I}\{\zeta\}| < \mathscr{R}\{k\}$ (see figure 1), and ζ is located below the path. The terms in (57) tend to zero when $|\zeta| \to \infty$ in the lower half-plane, as do the corresponding terms at infinity in the upper half-plane, and thus the integral function vanishes, yielding

$$\sqrt{(\zeta-ik)}\bar{F}_1(\zeta)+\frac{1}{2\pi i}\int_{C_-}\sqrt{(\xi-ik)}e^{2i\xi a}\bar{F}_2(\xi)\frac{d\xi}{\xi-\zeta}+$$
$$+\frac{1}{2\pi i}\int_{C_-}\sqrt{(\xi-ik)}\left(\frac{e^{2i\xi a}-1}{i\xi}\right)\frac{d\xi}{\xi-\zeta} = 0. \quad (58)$$

Similar considerations applied to (56) give

$$\sqrt{(\zeta+ik)}\bar{F}_2(\zeta)+\frac{1}{2\pi i}\int_{C_+}\sqrt{(\xi+ik)}e^{-2i\xi a}\,\bar{F}_1(\xi)\frac{d\xi}{\xi-\zeta}+$$
$$+\frac{1}{2\pi i}\int_{C_+}\sqrt{(\xi+ik)}\left(\frac{1-e^{-2i\xi a}}{i\xi}\right)\frac{d\xi}{\xi-\zeta}=0, \quad (59)$$

where C_+ is another path conducted in the strip, this time below the point ζ.

The integrals on C_-, C_+ are usefully recast by deforming the contours in the upper and lower half-planes, respectively, so as to run along the branch cuts drawn from $\zeta = \pm ik$. If these cuts are placed on the imaginary axis away from the origin ($\mathscr{I}\{k\}\to 0$), then, taking account of the phases of the radicals at each side of a cut, it follows that

$$\begin{aligned}\sqrt{(\zeta-ik)}\bar{F}_1(\zeta) &= -\frac{e^{i\pi/4}}{\pi}k^{1/2}\int_1^\infty\sqrt{(u-1)}\left(\frac{e^{-2kau}-1}{u}\right)\frac{du}{iku-\zeta}+\\&\quad+\frac{e^{i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\bar{F}_2(iku)e^{-2kau}\frac{du}{iku-\zeta}\\&=\frac{e^{i\pi/4}}{\pi}k^{1/2}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{iku-\zeta}+\\&\quad+\frac{e^{i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{iku-\zeta}, \quad (60)\end{aligned}$$

and

$$\begin{aligned}\sqrt{(\zeta+ik)}\bar{F}_2(\zeta) &= \frac{e^{3i\pi/4}}{\pi}k^{1/2}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{iku+\zeta}+\\&\quad+\frac{e^{3i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_1(-iku)-\frac{1}{ku}\right)e^{-2ku}\frac{du}{iku+\zeta}. \quad (61)\end{aligned}$$

The integral equations (60), (61) are not independent, for reversal of the sign of ζ in (60), and the use of relation $\bar{F}_1(\zeta)=\bar{F}_2(-\zeta)$, leads to (61). Let $\zeta = ikv$ in (61); then

$$\begin{aligned}\sqrt{(v+1)}\bar{F}_2(ikv) &= \frac{1}{\pi k}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{u+v}+\\&\quad+\frac{1}{\pi}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{u+v}\\&=-\frac{1}{kv}+\frac{\sqrt{(v+1)}}{kv}+\\&\quad+\frac{1}{\pi}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{u+v}, \quad (62)\end{aligned}$$

which is an integral equation for the function $\bar{F}_2(ikv)$. When $a\to\infty$, the integral drops out, and the resulting form of $\bar{F}_2$ agrees with that found in the semi-infinite strip configuration, namely (36). A more symmetrical version of the integral equation is obtained by defining

$$P(v)=\frac{1}{v}-k\,\bar{F}_2(ikv), \quad (63)$$

for then

$$\sqrt{(v+1)}P(v) = \frac{1}{v} + \frac{1}{\pi}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u+v}. \tag{64}$$

From (62) it also follows that

$$\bar{F}_1(0) = \bar{F}_2(0) = \frac{1}{2k} - \frac{1}{\pi k}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u}. \tag{65}$$

The relations (64), (65) provide a suitable basis for calculating the skin friction in the early stages of the motion, as will now be verified.

4. Skin friction

To begin the skin friction calculation, an expression for $\bar{g}(0)$ is required. Using (54), (65), this is given by

$$\bar{g}(0) = 4ka+4k\,\bar{F}_1(0) = 4ka+2-\frac{4}{\pi}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u}, \tag{66}$$

where $P(u)$ satisfies the integral equation (64). An iterative procedure can be applied to the integral equation when $\mathscr{R}\{k\} > 0$, and the leading terms of $P(u)$ thus obtained are

$$P(u) = \frac{1}{u\sqrt{(u+1)}} + \frac{1}{\pi\sqrt{(u+1)}}\int_1^\infty \sqrt{\left(\frac{v-1}{v+1}\right)}\frac{e^{-2kav}}{v}\frac{dv}{u+v}. \tag{67}$$

The corresponding approximation for $\bar{g}(0,p)$ is

$$\bar{g}(0,p) = 4a\sqrt{\left(\frac{p}{\nu}\right)}+2-\frac{4}{\pi}\int_1^\infty \sqrt{\left(\frac{u-1}{u+1}\right)}e^{-2a+\sqrt{(p/\nu)}u}\frac{du}{u^2} -$$
$$-\frac{4}{\pi^2}\int_1^\infty \sqrt{\left(\frac{(u-1)(v-1)}{(u+1)(v+1)}\right)}e^{-2a\sqrt{(p/\nu)}(u+v)}\frac{du\,dv}{uv(u+v)}, \tag{68}$$

after reverting to the Laplace transform variable,

$$p = \nu k^2. \tag{69}$$

There remains only an evaluation of the inversion integral (28),

$$D = \frac{\mu V}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\bar{g}(0,p)\frac{dp}{p} \quad (c > 0), \tag{70}$$

and it will be noted that the conditions $\mathscr{R}\{p\} > 0$, $\mathscr{R}\{k\} > 0$ are indeed compatible with (69). The resulting skin friction is

$$D = 2\mu V\left[\frac{2a}{\sqrt{(\pi\nu t)}}+1-\frac{2}{\pi}\int_1^\infty \sqrt{\left(\frac{u-1}{u+1}\right)}\operatorname{erfc}\left(\frac{au}{\sqrt{(\nu t)}}\right)\frac{du}{u^2} -\right.$$
$$\left.-\frac{2}{\pi^2}\int_1^\infty \sqrt{\left(\frac{(u-1)(v-1)}{(u+1)(v+1)}\right)}\operatorname{erfc}\left(\frac{a(u+v)}{\sqrt{(\nu t)}}\right)\frac{du\,dv}{uv(u+v)}\right], \tag{71}$$

where

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\xi^2}\,d\xi \tag{72}$$
$$\sim e^{-z^2}/\sqrt{\pi}z \quad \text{for } z \gg 1$$

is the complementary error function.

When $a/\sqrt{(\nu t)} \gg 1$, the principal contributions to the integrals arise from the neighbourhood of the lower limit, and thus the double integral is smaller than the single integral by a factor $e^{-3a^2}/\nu t$. This exponential decrease in magnitude evidently characterizes further contributions to the skin friction resulting from continued iteration of the integral equation. Successive terms acquire comparable magnitude when $a/\sqrt{(\nu t)} \ll 1$, and, in the limit $a/\sqrt{(\nu t)} \to 0$, the full iterated series is required to compensate the time independent term in D. The development (71) is therefore useful only for the early stages of the motion, as anticipated.

The first term in (71) arises from a uniform, local skin friction characteristic of an infinite strip, and the next term takes account of the non-uniformity introduced by the strip edges, here computed additively when the latter are infinitely remote. To verify this interpretation, observe that the excess skin friction on the semi-infinite strip $0 < x < \infty$, attributable to the edge, stems from the distribution (see (39))

$$g(x) = \frac{2k}{\pi}\int_1^\infty \sqrt{(u-1)}e^{-kxu}\frac{du}{u} \quad (x > 0), \tag{73}$$

whose integrated value over the range $0 < x < \infty$ is unity. The second term in (68), and hence (71), thus bespeaks two such contributions, from remote edges. If the distribution (73) is integrated over a strip $0 < x < 2a$, there is an additional contribution which is evidently distinct from the third term of (68). The necessary refinement of $g(x)$ entails a direct consideration of the finite strip configuration, and it is of interest to see how this may be carried out in the transform formulation.

In $|x| < a$, by (24),

$$-\tfrac{1}{2}g(x) = \frac{1}{2\pi}\left(\frac{d^2}{dx^2}-k^2\right)\left[\int_{-\infty}^{\infty} e^{i\zeta x}\frac{2\sin\zeta a}{\zeta}\frac{d\zeta}{\sqrt{(k^2+\zeta^2)}} + \right.$$
$$\left. + \int_{-\infty}^{\infty} e^{i\zeta x}\{e^{-i\zeta a}\bar{F}_1(\zeta)+e^{i\zeta a}\bar{F}_2(\zeta)\}\frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}\right], \tag{74}$$

$$\bar{F}_1(\zeta) = -\frac{i}{\zeta}\left\{1-\sqrt{\left(\frac{-ik}{\zeta-ik}\right)}\right\} -$$
$$-\frac{e^{i/\pi 2}}{\pi}\sqrt{\left(\frac{-ik}{\zeta-ik}\right)}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{iku-\zeta} = \bar{F}_2(-\zeta). \tag{75}$$

A modification of the integrals is expedient, and results from deforming the contour into the upper or lower part of the ζ-plane. Thus, for $|x| < a$,

$$\int_{-\infty}^{\infty} e^{i\zeta x}\frac{2\sin\zeta a}{\zeta}\frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}$$
$$= \frac{1}{2i}\int_{-\infty}^{\infty}\{(1-e^{-i\zeta(a+x)})-(1-e^{i\zeta(a+x)})+$$
$$+(1-e^{-i\zeta(a-x)})-(1-e^{i\zeta(a-x)})\}\frac{d\zeta}{\zeta\sqrt{(k^2+\zeta^2)}}$$
$$= \frac{2}{k}\int_1^\infty (2-e^{-k(a+x)u}-e^{-k(a-x)u})\frac{du}{u\sqrt{(u^2-1)}}, \tag{76}$$

and

$$\int_{-\infty}^{\infty} e^{-i\zeta(a-x)}\bar{F}_1(\zeta)\frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}$$

$$= \frac{2}{k}\int_1^{\infty}\frac{e^{-k(a-x)u}}{u}\left(1-\frac{1}{\sqrt{(u+1)}}\right)\frac{du}{\sqrt{(u^2-1)}}-$$

$$-\frac{2}{\pi k}\int_1^{\infty}\frac{e^{-k(a-x)u}}{\sqrt{(u+1)}}\left\{\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{\sqrt{(u^2-1)}}, \quad (77)$$

while the corresponding integral with $\bar{F}_2(\zeta)$ calls only for a reversal in the sign of x in the above expressions. Substituting these expressions in (74), we obtain

$$g(x) = 2k+\frac{2}{\pi k}\left(\frac{d^2}{dx^2}-k^2\right)\left[\int_1^{\infty}\frac{e^{-k(a+x)u}+e^{-k(a-x)u}}{\sqrt{(u+1)}}\times\right.$$

$$\left.\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{\sqrt{(u^2-1)}}\right]$$

$$=2k+\frac{2k}{\pi}\left[\int_1^{\infty}\sqrt{(u-1)}(e^{-k(a+x)u}+e^{-k(a-x)u})\times\right.$$

$$\left.\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}du\right]$$

$$= 2k+\frac{2k}{\pi}\int_1^{\infty}\sqrt{(u^2-1)}(e^{-k(a+x)u}+e^{-k(a-x)u})P(u)\,du, \quad (78)$$

where the last form follows from a use of the integral equation for $P(u)$. In the second form, there may be recognized contributions of the type (73), referred to each strip edge. The singularities of $g(x)$ at $x = \pm a$ are contained in factors $(a\pm x)^{-1/2}$, which are also characteristic of an isolated edge. The time varying function $g(x, t)$ obtains after the inverse Laplace transform is applied to (78), and, with the first approximation $P(u) = 1/u\sqrt{(u+1)}$, the result is

$$g(x, t) = \frac{2}{\sqrt{(\pi\nu t)}}+\frac{2}{\pi\sqrt{(\pi\nu t)}}\int_1^{\infty}\sqrt{(u-1)}\times$$

$$\times\left\{\exp\left(-\frac{(a+x)^2u^2}{4\nu t}\right)+\exp\left(-\frac{(a-x)^2u^2}{4\nu t}\right)\right\}\frac{du}{u}. \quad (79)$$

After integration of (78), one finds

$$\int_{-a}^{a} g(x)\,dx = 4ka+\frac{4}{\pi}\int_1^{\infty}\sqrt{(u-1)}(1-e^{-2kau})\times$$

$$\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{u} \quad (80\text{ a})$$

$$= 4ka+\frac{4}{\pi}\int_1^{\infty}\sqrt{(u^2-1)}(1-e^{-2kau})P(u)\frac{du}{u}, \quad (80\text{ b})$$

in contrast with (66). However, on substitution of the iterative expression for $P(u)$, the same development is obtained with each of these representations; it is noteworthy that (66) and (80 a) lead to the result (68) when $P(u)$ is given by the two-term approximation (67), whereas a three-term approximation is called for in (80 b). This feature reflects a difference in basis of the calculation according as the off-strip distribution $f(x)$ for $|x| > a$, or the strip distribution $g(x)$ for $|x| < a$, is involved. The semi-infinite distribution is a natural first approximation for $f(x)$, and each successive stage of approximation generates a distinct order-of-magnitude correction in both $f(x)$ and the skin friction D. On the other hand, only a finite part of the semi-infinite distribution for $g(x)$ is relevant, and successive approximations are partially coupled in magnitude, resulting in like order contributions to D.

This work was supported by the office of Naval Research.

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